1. The curve C has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \quad 0 \leqslant t \leqslant \pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(2)

The point P lies on C where $t = \frac{2\pi}{3}$

The line l is the normal to C at P.

(b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q.

(c) Find the exact coordinates of Q.

You must show clearly how you obtained your answers.

(6)

a) $x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$ $\qquad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ 1

$\frac{dx}{dt} = -2\sin t$ $\qquad \frac{dy}{dt} = 2 \times \sqrt{3} \times -\sin(2t)$

$\frac{dy}{dt} = -2\sqrt{3}\sin(2t)$ $\qquad \sin(2t) = 2\sin t\cos t$

$= \frac{dy}{dx} = \frac{-2\sqrt{3}\sin(2t)}{-2\sin(t)} = \frac{\sqrt{3}(2\sin t\cos t)}{\sin(t)} = \frac{2\sqrt{3}\cancel{\sin t}\cos t}{\cancel{\sin} t}$

$\Rightarrow \frac{dy}{dx} = \underline{2\sqrt{3}\cos(t)}$ 1

Question 1 continued

b) $\frac{dy}{dx} = 2\sqrt{3}\cos(t) = 2\sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\sqrt{3}$ (1)

Gradient of the normal $= -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \frac{1}{\sqrt{3}} = m$ (1)

$t = \frac{2\pi}{3}$ and $x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$

$\Rightarrow x = 2\cos\left(\frac{2\pi}{3}\right)$ $\quad y = \sqrt{3}\cos\left(2 \times \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\Rightarrow x = -1$ (1)

$y - \left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{3}}\left(x - (-1)\right) \Rightarrow y = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2}$ (1)

$\Rightarrow y = \frac{1}{\sqrt{3}}x - \frac{\sqrt{3}}{6}$

$\times\sqrt{3}$ $\Rightarrow \sqrt{3}y = x - \frac{1}{2}$

$\times 2$ $\Rightarrow 2\sqrt{3}y = 2x - 1$

$\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ as required. (1)

c) $x = 2\cos t \quad y = \sqrt{3}\cos(2t)$

Eq of line l: $2x - 2\sqrt{3}y - 1 = 0$

$\Rightarrow 2(2\cos t) - 2\sqrt{3}\left(\sqrt{3}\cos(2t)\right) - 1 = 0$ (1) $\quad 6\cos(2t) = 6(2\cos^2 t - 1)$

$\Rightarrow 4\cos t - 6\cos(2t) - 1 = 0$ $\quad = 12\cos^2 t - 6$

$\times -1$ $\Rightarrow 4\cos t - 12\cos^2 t + 6 - 1 = 0$ (1)

$\Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$ (1)

Now, let $\theta = \cos t$

$12\theta^2 - 4\theta - 5 = 0 \Rightarrow \theta = \frac{4 \pm \sqrt{(-4)^2 - 4(12)(-5)}}{2 \times 24}$

+ve $\sqrt{\ }$: $\theta = \frac{5}{6}$, -ve $\sqrt{\ }$: $\theta = -\frac{1}{2}$

$\cos t = \frac{5}{6}$ $\quad \cos t = -\frac{1}{2} \Rightarrow t = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \Rightarrow$ ignore solution. (1)

$x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$ $\quad 2t = \cos^{-1}(5/6) \times 2$

$x = 2 \times \frac{5}{6} = \frac{5}{3}$ $\quad y = \sqrt{3}\cos\left(\cos^{-1}(5/6) \times 2\right) = \frac{7\sqrt{3}}{18}$ (1)

$Q : \left(\frac{5}{3}, \frac{7}{18}\sqrt{3}\right)$ (1)

2. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

1 separate terms:

a) $\frac{d}{dx}(3y^2) = 6y\frac{dy}{dx}$

$\frac{d}{dx}(qxy) = qx\frac{dy}{dx} + qy$ ← apply the product rule.

when $y = u(x)v(x)$, $\frac{dy}{dx} = \frac{du}{dx}v + \frac{dv}{dx}u$

$\frac{d}{dx}(px^3) = 3px^2$

$3px^2 + qx\frac{dy}{dx} + qy + 6y\frac{dy}{dx} = 0$ 1

rearrange to make $\frac{dy}{dx}$ the subject

$(qx + 6y)\frac{dy}{dx} = -3px^2 - qy$ 1

$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}$ 1

b) when $x = -1$ and $y = -4$:

$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$ 1 use original curve to make first equation

$-p + 4q + 48 = 26$

$4q - p = -22$ 1

DO NOT WRITE IN THIS AREA

Question 2 continued

$19x + 26y + 123 = 0$

$26y = -19x - 123$

$y = -\frac{19}{26}x - \frac{123}{26}$ $\therefore m = -\frac{19}{26}$ ① rearrange normal equation to find gradient

$\frac{dy}{dx} = m$ at $(-1, -4)$ gradients are equal.

$$\frac{-3px^2 - qy}{qx + 6y} = -\frac{19}{26} \quad ①$$

$$\frac{-3p(-1)^2 - q(-4)}{q(-1) + 6(-4)} = -\frac{19}{26}$$ substitute in $(-1, -4)$

simplify to make second equation.

$57p - 102q = 624$ ② ①

solve ① and ② simultanously to give:

solve simultaneously (by hand or using a calculator)

$p = 2,\ q = -5$ ①

3. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3\theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)

a) $y = \operatorname{cosec}^3\theta$

$y = (\operatorname{cosec}\theta)^3$

using product rule for brackets
$\frac{d}{dx}(f(x))^n = f'(x) \times (n-1) \times (f(x))^{n-1}$

$\frac{dy}{d\theta} = 3 \times -\operatorname{cosec}\theta\cot\theta \times (\operatorname{cosec}\theta)^2 = -3\operatorname{cosec}^2\theta\operatorname{cosec}\theta\cot\theta$ 1

$= -3\operatorname{cosec}^3\theta\cot\theta$

$x = \sin 2\theta \qquad \frac{dx}{d\theta} = 2\cos 2\theta$ using differentiation laws for trig

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ 1

$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$ 1

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

b) $y = 8 \Rightarrow \operatorname{cosec}^3\theta = 8 \Rightarrow \sin^3\theta = \frac{1}{8}$ 1

$\sin\theta = \sqrt[3]{\frac{1}{8}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ remember to use radians!

$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right)}{2\cos\left(\frac{2\pi}{6}\right)}$ 1 (put in calculator)

$\frac{dy}{dx} = -24\sqrt{3}$ 1